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The Heat Kernel on the Two-Sphere

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An explicit full solution to the heat equation on the two-sphere is given.

0. INTRODUCTION

In this paper we give an explicit solution, $E(t, gH)$, for the heat equation (with initial data concentrated at a point) on S^2 regarded as the homogeneous space $G/H = SU(2)/U(1)$. For initial data concentrated at the identity coset our solution is

$$E(t, gH) = \sum_{n=0}^{\infty} (2n+1)\, e^{-n(n+1)t/2} P_n\left(\tfrac{1}{2}\{(\operatorname{tr} g)^2 + (\operatorname{tr} i_0 g)^2\} - 1\right) \qquad (1)$$

where $P_n(x)$ are the Legendre polynomials, "tr" means trace, and $i_0 \in SU(2)$ is the matrix $\begin{pmatrix} i & \\ & -i \end{pmatrix}$, where $i = \sqrt{-1}$.

Previously solutions have been computed (1) on S^2, but only at the identity coset, i.e., only $E(t, H)$ was computed (Benabdallah [1], Cahn and Wolf [2]); (2) fully, but only for compact semi-simple Lie groups (Fegan [5]); or (3) for other types of homogeneous spaces (Eskin [4]).

1. THE METHOD

Our principal tool is the following formula of Benabdallah [1] which holds for an arbitrary compact homogeneous space G/H:

$$E_{G/H}(t, gH) = \sum_{\alpha \in \Lambda_H} d_\alpha e^{-\lambda_\alpha t} \phi_\alpha(gH) \qquad (2)$$

where (1) Λ_H is an index set for the equivalence classes of irreducible unitary representations π_α of G which are *class* 1 with respect to H (see Section 3 for the definition), (2) d_α is the dimension of the representation space of π_α, (3) $\{\lambda_\alpha\}$ is a subset of the eigenvalues (with multiplicity) of the Laplacian on

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G (we make this more precise in Section 3), and (4) $\phi_\alpha(gH) = \int_H x_\alpha(gH)\,dh$ where χ_α is the character of π_α and dh is normalized Haar measure on H.

For $G = SU(2)$ the data (1), (2), and (3) are well known (see Section 3 below). Our contribution is to compute (4).

In speaking of a "heat equation" on a manifold one must specify which heat equation, i.e., what choice of Laplacian is being made. We specify that the Laplacian on $S^2 \approx SU(2)/U(1)$ is the Laplace–Beltrami operator associated to the left-invariant Riemannian structure inherited from that Riemannian structure on $SU(2)$ derived from the negative of the Cartan–Killing form on $su(2)$, the Lie algebra of $SU(2)$.

The form of the heat equation considered is

$$\begin{aligned} &\Delta E(t, gH) + \frac{\partial}{\partial t} E(t, gH) = 0 \\ &\lim_{t \to 0+} \int_{G/H = S^2} E(t, gH) f(gH) = f(eH), \qquad \text{for all } f \in C^\infty(G/H). \end{aligned} \tag{3}$$

The usual heat equation over Euclidean domains has the form "$\Delta - \partial/\partial t$" $= 0$. The reason for the discrepancy in sign is that, in our case, the Laplacian, by construction, will have positive eigenvalues; whereas in the Euclidean case, the usual Δ has, strictly speaking, eigenvalues that are negative. Our solution (1) may be regarded as an extension, involving special functions, of the method of separation of variables to a non-Euclidean domain.

2. The Compact Homogeneous Case

Assume that a manifold M comes equipped with a Laplacian. Then the heat kernel on M is a function satisfying the two equations

$$\Delta_x E_M(t; x, y) + \frac{\partial}{\partial t} E_M(t; x, y) = 0$$

and (4)

$$\phi(y) = \lim_{t \to 0+} \int_M E_M(t; x, y)\, \phi(x)\, dx.$$

Then $E_M(t; \cdot, y)$ is the solution to the heat equation with initial concentration at point y.

When M is a homogeneous space G/H with G compact, then there is a natural bi-invariant Riemannian metric on G. By left translation on the quotient, one obtains a left-invariant Riemannian structure on M. With such a choice. Benabdallah [1]

(a) shows that $E_{G/H}$ is left invariant (i.e., $E_{G/H}(t; zH, wH) = E_{G/H}(t; gzH, gwH)$, so that $E_{G/H}(t; zH, wH) = E_{G/H}(t; w^{-1}zH, H)$) and we may thus write $E_{G/H}(t; gH, H) = E_{G/H}(t, gH)$; and

(b) obtains his formula, (2).

When, in addition, G is semi-simple we may make a canonical choice of such a bi-invariant Riemannian structure on G, namely, that deriving from the negative of the Cartan–Killing form of the Lie algebra of G. (The "negative" is there in order to have a positive definite form.) This choice is particularly useful; for then we can regard Δ as the Casimir operator and compute eigenvalues of Δ—which appear in the formula (2)—as eigenvalues of the Casimir, via the representation theory of G.

$E_{G/H}(t, gH, g_1H)$ represents the value at point gH and time t of the solution to the heat equation with initial concentration at g_1H. By left invariance, $E_{G/H}(t; gH, g_1H) = E_{G/H}(t; g_1^{-1}gH, H) = E_{G/H}(t, g_1^{-1}gH)$; i.e., the "initial data at g_1H" solution is the "initial data at identity coset" solution translated by g_1^{-1}.

3. Results from the Representation Theory of $SU(2)$

Any $g \in SU(2)$ can be expressed as $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, where α, β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. We regard $U(1) \approx S^1$ as imbedded in $SU(2)$ under the map

$$e^{i\theta} \mapsto h_\theta = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}.$$

An irreducible representation $\pi : G \to \mathrm{Aut}(V)$ of G is called *Class* 1 with respect to a closed subgroup H of G if the subrepresentation of π obtained by restriction to H leaves fixed a non-zero vector of V, i.e., if there exists $v \in V$, $v \neq 0$, such that $\pi(h)\, v = v$ for all $h \in H$.

We now assemble the facts that we need from the representation theory of $SU(2)$:

(a) For each integer $m \geqslant 0$, there is an irreducible representation π_m of $SU(2)$ with representation space of dimension $m + 1$.

(b) These are, up to equivalence, all of the irreducible unitary representations of $SU(2)$.

(c) Let χ_m be the character of π_m. Then

$$\chi_m(h_\phi) = \frac{e^{i(m+1)\phi} - e^{-i(m+1)\phi}}{e^{i\phi} - e^{-i\phi}} = e^{im\phi} + e^{i(m-2)\phi} + \cdots + e^{-im\phi}. \tag{5}$$

(d) The *Class* 1 representations are precisely those for which $m = 2n$ is even, hence have representation spaces of dimension $2n + 1$. And thus when $m = 2n$ (5) says that

$$\chi_m(h_\phi) = 1 + \sum_{k=1}^{n} (e^{i2k\phi} + e^{-i2k\phi}) = 1 + 2 \sum_{k=1}^{n} \cos 2k\phi. \qquad (6)$$

(e) Because of our choice of Riemannian structure on $G = SU(2)$, Δ_G can be regarded as the Casimir operator on G. On a compact Lie group, the characters of the irreducible unitary representations are *eigenfunctions* of the Casimir. For $G = SU(2)$ the eigenvalue λ_m of the character χ_m is known to be $m(m+2)/8$.

Thus when π_m is *Class* 1 with respect to $H = U(1)$, so that $m = 2n$, then $\lambda_m = \lambda_{2n} = n(n+1)/2$. That is, $\{\lambda_{2n}\}$ is the subset of those eigenvalues of the Laplacian on $G = SU(2)$ that constitute the eigenvalues of the Laplacian on $G/H = S^2$.

Given (a)–(e) formula (2) becomes

$$E_{S^2}(t, gH) = \sum_{n=0}^{\infty} (2n+1)\, e^{-n(n+1)t/2} \phi_{2n}(gH); \qquad (7)$$

and it remains to compute $\phi_{2n}(gH)$, which is done in Sections 4 to 6.

4. Reduction to Definite Integrals

Let $m = 2n$. Now $\phi_m(gH) = \int_H \chi_m(gh)\, dh = 1/2\pi \int_{\phi=0}^{2\pi} \chi_m(gh_\phi)\, d\phi$, as the normalized Haar measure dH on $U(1) \approx S^1$ is given by $1/2\pi\, d\phi$. Since $H = U(1)$ is a maximal torus of $SU(2)$ we can always find $a \in SU(2)$ (depending on g and ϕ) such that $a^{-1} g h_\phi a \in U(1)$, i.e., $a^{-1} g h_\phi a = h_\eta$ for some η. Thus h_η is the diagonalization of gh_ϕ. This is useful as $\chi_m(h_\eta) = \chi_m(a^{-1} g h_\phi a) = \chi_m(gh_\phi)$, and χ_m is more easily computed on torus elements. In fact (from (6))

$$\chi_m(h_\eta) = 1 + 2 \sum_{k=1}^{n} \cos 2k\eta = 1 + 2 \sum_{k=1}^{n} Q_k(\cos^2 \eta) \qquad (8)$$

as $\cos 2k\eta$ may be expressed as some polynomial expression Q_k in the argument $\cos^2 \eta$. Thus we can integrate $\chi_m(gh_\phi)$ if we can relate η to ϕ.

Lemma. *Let* $g = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} \in SU(2)$ *and write* $\cos \eta = \mathrm{Re}(\alpha)$; *then the eigenvalues of* g *are* $e^{\pm i\eta}$. *In particular* g *diagonalizes as the matrix* h_η.

Proof. This is obtained by solving for λ in the equation $\det(g - \lambda I) = 0$, recognizing that $|\alpha|^2 + |\beta|^2 = 1$.

Write $\alpha = |\alpha| e^{i\theta}$; then

$$gh_\phi = \begin{pmatrix} |\alpha| e^{i(\phi+\theta)} & * \\ * & * \end{pmatrix}.$$

Thus by the lemma applied to gh_ϕ, η is related to ϕ by $\cos\eta = \mathrm{Re}(|\alpha| e^{i(\phi+\theta)}) = |\alpha| \cos(\phi+\theta)$. So

$$\begin{aligned}\phi_m(gH) &= \int_H \chi_m(gh)\,dh = \frac{1}{2\pi}\int_{\phi=0}^{2\pi} \left\{1 + 2\sum_{k=1}^{n} Q_k(\cos^2\eta)\right\} d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\{1 + 2\sum_{k=1}^{n} Q_k(|\alpha|^2\cos^2(\phi+\theta))\right\} d\phi.\end{aligned} \tag{9}$$

Since the integrand in (9) is 2π-periodic while the integration is over a full period we may make the invariant transformation $\phi \mapsto \phi - \theta$, thus getting

$$\phi_m(gH) = \frac{1}{2\pi}\int_0^{2\pi} \left\{1 + 2\sum_{k=1}^{n} Q_k(|\alpha|^2\cos^2\phi)\right\} d\phi \tag{10}$$

an integral in the argument ϕ alone.

Consulting Gradshteyn and Ryzhik [6], we find a formula for Q_k (p. 27, no. 1.331.2, 2nd form) and a formula for the definite integrals of even powers of cos (p. 369, no. 3.621.3). Using these plus a slight rearrangement of factorial expressions appearing in the binomial coefficients involved in these formulae one arrives at the following:

$$\begin{aligned}\phi_m(gH) = (-1)^n + 2\sum_{k=1}^{n} &\left\{\binom{2k-1}{k}|\alpha|^{2k}\right. \\ &\left. + \sum_{j=1}^{k-1} (-1)^j \frac{2k}{j}\binom{2k-j-1}{j-1}\binom{2k-2j-1}{k-j}|\alpha|^{2k-2j}\right\}\end{aligned} \tag{11}$$

where, when $k=j$, $\binom{-1}{0}$ is taken to mean the value $\frac{1}{2}$. In Section 5 below we simplify this expression.

5. Combinatorial Reduction

In (11) collect terms involving like powers of $|\alpha|$, substituting $p = k - j$. Equation (11) becomes

$$\begin{aligned}\phi_m(gH) = (-1)^n + 2\sum_{p=1}^{n} &\binom{2p-1}{p} \\ &\times \left\{1 + \sum_{k=p+1}^{n} (-1)^{k-p}\frac{2k}{k-p}\binom{k+p-1}{k-p-1}\right\}|\alpha|^{2p}.\end{aligned} \tag{12}$$

In (12) note that

$$\frac{2k}{k-p}\binom{k+p-1}{k-p-1}=\frac{k}{p}\binom{k+p-1}{k-p}$$

and make the substitution $r=k-p$. Then the coefficient of $|\alpha|^{2p}$ $(1\leqslant p\leqslant n)$ in (12) becomes

$$\binom{2p}{p}\left\{1+\sum_{r=1}^{n-p}(-1)^r\frac{r+p}{p}\binom{r+2p-1}{r}\right\}. \tag{13}$$

We focus now just on that part of (13) that involves the sum of the binomial coefficients that appear there. By rearranging factorials we have

$$\frac{r+p}{p}\binom{r+2p-1}{r}=\binom{r+2p}{r}+\binom{r+2p-1}{r-1}. \tag{14}$$

Summing the right side of (14) from $r=1$ to $r=n-p$, to form the summation appearing in (13), while substituting s for $r-1$ in the sum of the second of the two binomial terms appearing on the right side of (14), we can observe a telescopic cancellation: the summation appearing in (13) reduces to $(-1)^{n-p}\binom{n+p}{n-p}-1$. Consequently the coefficient in (12) of $|\alpha|^{2p}$, expressed as (13), becomes simply

$$(-1)^{n-p}\binom{2p}{p}\binom{n+p}{n-p}. \tag{15}$$

Then (12) and (15) together yield

$$\phi_m(gH)=\sum_{p=0}^{n}(-1)^{n-p}\binom{2p}{p}\binom{n+p}{n=p}|\alpha|^{2p},$$

which can be rewritten in equivalent form as

$$\phi_m(gH)=(-1)^n\sum_{p=0}^{n}\binom{n}{p}\binom{n+p}{p}(-|\alpha|^2)^p. \tag{16}$$

6. Orthogonal Polynomials

The Jacobi polynomials $P_n^{(a,b)}(x)$ are defined (compare with Erdélyi *et al.* [3, Vol. 2, p. 170, (16), and Vol. 1, p. 101, (1)]; then rearrenge factors), for $a,b>-1$, as

$$P_n^{(a,b)}(x)=\binom{n+a}{n}\sum_{p=0}^{n}\frac{\binom{n}{p}\binom{n+a+b+p}{p}}{\binom{a+p}{p}}\left(\frac{x-1}{2}\right)^p. \tag{17}$$

Hence when $a=b=0$ and $x=1-2|\alpha|^2$ it is clear from (16) and (17) that $\phi_m(gH)=(-1)^n P_n^{(0,0)}(1-2|\alpha|^2)=P_n^{(0,0)}(2|\alpha|^2-1)$ (see [3, Vol. 2, p. 170]). In the special case of indices (0, 0), $P_n^{(0,0)}(x)=P_n(x)$, the nth Legendre polynomial (see [6, p. 1036, no. 8, 962.2]). Thus $\phi_m(gH)=P_n(2|\alpha|^2-1)$.

It remains to express $|\alpha|^2$ as a function of g. Let

$$i_0=\begin{pmatrix} i & \\ & -i \end{pmatrix} \in SU(2).$$

Then it is easy to check that $\operatorname{tr} g = 2\operatorname{Re}(\alpha)$ and $\operatorname{tr} i_0 g = -2\operatorname{Im}(\alpha)$, so that $2|\alpha|^2-1=\frac{1}{2}\{(\operatorname{tr} g)^2+(\operatorname{tr} i_0 g)^2\}-1$. Hence

$$\phi_m(gH)=P_n(\tfrac{1}{2}\{(\operatorname{tr} g)^2+(\operatorname{tr} i_0 g)^2\}-1). \tag{18}$$

Putting this into (7), we finally obtain our solution (1).

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